E2.5 Signals & Linear Systems

Tutorial Sheet 6 – Solutions

1\. a)

$$F(\omega) = \int_0^T e^{-at} e^{-j\omega t}\, dt$$

$$= \int_0^T e^{-(a+j\omega)t}\, dt$$

$$= \frac{1 - e^{-(a+j\omega)T}}{a + j\omega}$$

b)

$$F(\omega) = \int_0^T e^{at} e^{-j\omega t}\, dt$$

$$= \int_0^T e^{-(-a+j\omega)t}\, dt$$

$$= \frac{1 - e^{-(-a+j\omega)T}}{-a + j\omega}$$

(Could make $a = -a$ and apply results in (a))

2\. a)

$$f(t) = \frac{1}{2\pi}\int_{-\omega_0}^{\omega} \omega^2 e^{j\omega t}\, d\omega$$

Don't forget we integrate wrt ω, NOT t.

$$= \frac{1}{2\pi}\, \frac{e^{j\omega t}}{(jt)^3}\left[-\omega^2 t^2 - 2j\omega t + 2\right]\Bigg|_{-\omega_0}^{\omega_0}$$

$$= \frac{(\omega_0^2 t^2 - 2)\sin\omega_0 t + 2\omega_0 t\cos\omega_0 t}{\pi t^3}$$

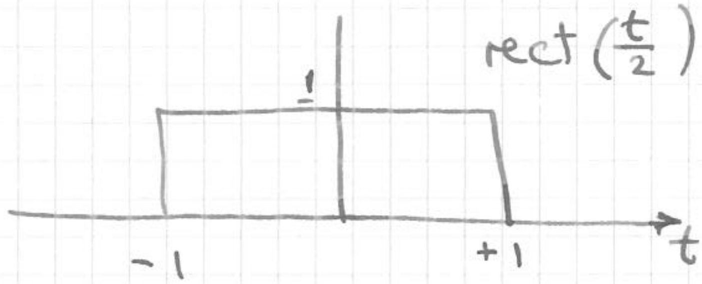
b)

$$f(t) = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \cos\omega\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{4\pi}\int_{-\pi/2}^{\pi/2} \left[e^{j\omega(1+t)} + e^{-j\omega(1-t)}\right] d\omega$$

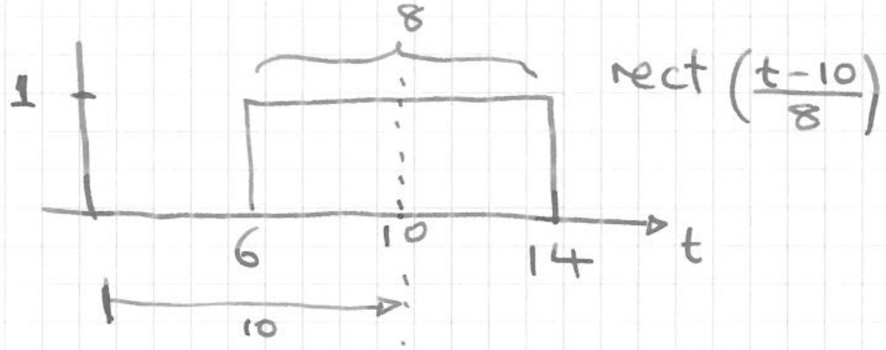
$$= \frac{1}{2\pi}\left[\frac{\sin (1+t)\pi/2}{1+t} + \frac{\sin(-1+t)\pi/2}{1+t}\right]$$

3. a)



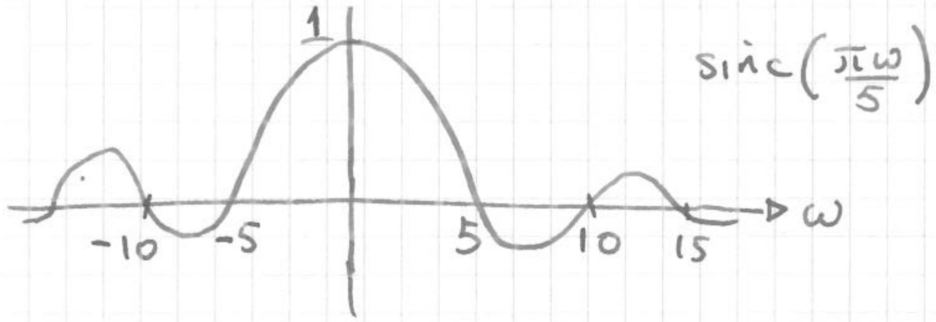
1
$\mathrm{rect}\left(\frac{t}{2}\right)$
-1
$+1$
t

b)



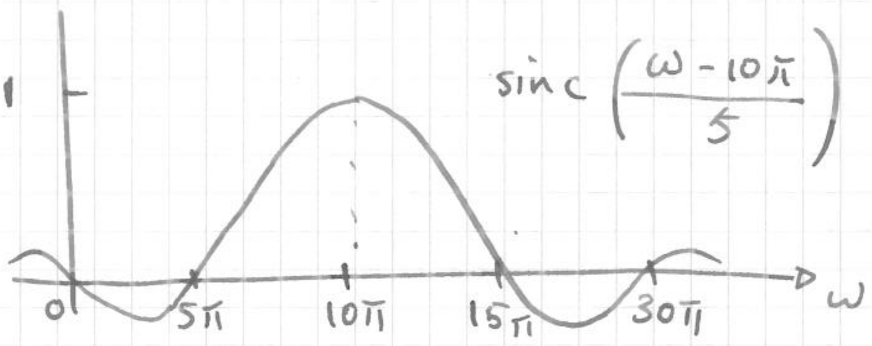
8
1
$\mathrm{rect}\left(\frac{t-10}{8}\right)$
6
10
14
t
10

c)



1
$\mathrm{sinc}\left(\frac{\pi\omega}{5}\right)$
-10
-5
5
10
15
ω

d)



1
$\mathrm{sinc}\left(\frac{\omega-10\pi}{5}\right)$
0
5π
10π
15π
30π
ω

4. a) From lec 10, slide 14. FT table
pair # 10,

$$\underbrace{u(t)}_{f(t)} \Longleftrightarrow \underbrace{\pi\delta(\omega) + \frac{1}{j\omega}}_{F(\omega)}$$

Duality property is given in Lect 10, slide 5.

Applying the duality property yields,

$$\underbrace{\pi\delta(t) + \frac{1}{jt}}_{F(t)} \Longleftrightarrow \underbrace{2\pi\, u(-\omega)}_{2\pi f(-\omega)}$$

$$\therefore \frac{1}{2}\left[\delta(t) + \frac{1}{j\pi t}\right] \Longleftrightarrow u(-\omega)$$

Apply scaling property $(a = -1)$,

$$\frac{1}{2}\left[\delta(-t) - \frac{1}{j\pi t}\right] \Longleftrightarrow u(\omega)$$

$$\delta(-t) = \delta(t)$$

$$\therefore \frac{1}{2}\left[\delta(t) + \frac{j}{\pi t}\right] \Longleftrightarrow u(\omega)$$

4. b) We need to show

$$\frac{1}{t} \Longleftrightarrow -j\pi \,\mathrm{sgn}(\omega)$$

(Error in question: "sng" should be "sgn")

From FT table, Lec 10/14, #12

$$\underbrace{\mathrm{sgn}(t)}_{f(t)} \Longleftrightarrow \underbrace{\frac{2}{j\omega}}_{F(\omega)}$$

∴ Using duality property:

$$\underbrace{\frac{2}{jt}}_{F(t)} \Longleftrightarrow \underbrace{2\pi \,\mathrm{sgn}(-\omega)}_{2\pi f(-\omega)} = -2\pi \,\mathrm{sgn}(\omega)$$

$$\therefore \quad \frac{1}{t} \Longleftrightarrow -j\pi \,\mathrm{sgn}(\omega) \quad //$$

c) Show $\delta(t+T) - \delta(t-T) \Longleftrightarrow 2j\sin(T\omega)$

From FT table, #10

$$\underbrace{\sin \omega_0 t}_{f(t)} \Longleftrightarrow j\pi \underbrace{[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]}_{F(\omega)}$$

$$\therefore \quad j\pi[\delta(t+\omega_0) - \delta(t-\omega_0)] \Longleftrightarrow 2\pi \sin(-\omega_0\omega)$$

$$= -2\pi \sin(\omega_0 \omega)$$

Let $\omega_0 = T$ yields

$$j\pi[\delta(t+T) - \delta(t-T)] \Longleftrightarrow -2\pi \sin(\omega T)$$

$$\therefore \quad \delta(t+T) - \delta(t-T) \Longleftrightarrow 2j \sin(\omega T) \quad //$$

5. Fig (b) $f_1(t) = f(-t)$

$$\therefore F_1(\omega) = F(-\omega) = \frac{1}{\omega^2}\left[e^{-j\omega} + j\omega e^{-j\omega} - 1\right]$$

Fig (c) $f_2(t) = f(t-1) + f_1(t-1)$.

$$\therefore F_2(\omega) = \left[F(\omega)e^{-j\omega} + F_1(\omega)e^{-j\omega}\right]$$

$$= \left[F(\omega) + F(-\omega)\right]e^{-j\omega}$$

$$= \frac{2}{\omega^2}\left(\cos\omega + \omega\sin\omega - 1\right)$$

Fig (d) $f_3(t) = f(t-1) + f_1(t+1)$.

$$\therefore F_3(\omega) = F(\omega)e^{-j\omega} + F(-\omega)e^{j\omega}$$

$$= \frac{1}{\omega^2}\left[2 - 2\cos\omega\right]$$

$$= \frac{4}{\omega^2}\sin^2\frac{\omega}{2} = \mathrm{sinc}^2\left(\frac{\omega}{2}\right)$$

Fig (e) $f_4(t) = f(t-\tfrac{1}{2}) + f_1(t+\tfrac{1}{2})$

$$\therefore F_4(\omega) = \frac{e^{-j\omega/2}}{\cancel{\ }}\cancel{\ }F(\omega) + e^{j\omega/2}F(-\omega)$$

$$= \frac{e^{-j\omega/2}}{\omega^2}\left[e^{j\omega} - j\omega e^{j\omega} - 1\right] + \frac{e^{j\omega/2}}{\omega^2}\left[e^{-j\omega} + j\omega e^{-j\omega} - 1\right]$$

$$= \frac{1}{\omega^2}\left[2\omega\sin\frac{\omega}{2}\right]$$

$$= \mathrm{sinc}\left(\frac{\omega}{2}\right)$$

5) Fig (f) $f_5(t)$ can be constructed in 3 steps.

Step 1 : Time expand $f(t)$ by a factor of 2.

$$f\left(\frac{t}{2}\right) \Longleftrightarrow 2F(2\omega) = \frac{1}{2\omega^2}\left(e^{-j2\omega} - j2\omega e^{j2\omega} - 1\right)$$

Step 2 : Delay this by 2 seconds.

$$f\left(\frac{t-2}{2}\right) \Longleftrightarrow \frac{1}{2\omega^2}\left(e^{-j2\omega} - j2\omega e^{j2\omega} - 1\right) e^{-j2\omega}$$

$$= \frac{1}{2\omega^2}\left(1 - j2\omega - e^{-j2\omega}\right)$$

Step 3: Multiply result by 1.5 :

$$f_5(t) = 1.5\, f\left(\frac{t-2}{2}\right) \Longleftrightarrow \frac{3}{4\omega^2}\left(1 - j2\omega - e^{-j2\omega}\right)$$

//

6. (a) $f(t)$ is a triangular pulse $\Delta\left(\frac{t}{2\pi}\right)$ multiplied by $\cos 10t$.

$$f(t) = \Delta\left(\frac{t}{2\pi}\right)\cos 10t \, .$$

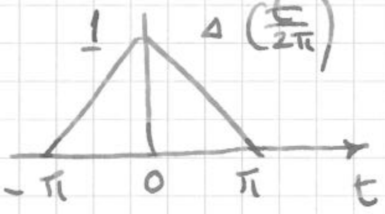
$\therefore$ From FT table pair # 19,

$$\Delta\left(\frac{t}{2\pi}\right) \iff \pi \,\text{sinc}^2 \frac{\pi\omega}{2} .$$

Using modulation property, it follows:

$$\Delta\left(\frac{t}{2\pi}\right)\cos 10t \iff \pi\left\{\text{sinc}^2\left[\frac{\pi(\omega-10)}{2}\right] + \text{sinc}^2\left[\frac{\pi(\omega+10)}{2}\right]\right\}$$

Since the signal is an even function, the Fourier transform is real, $\therefore$ needs only amplitude spectrum (phase is zero).



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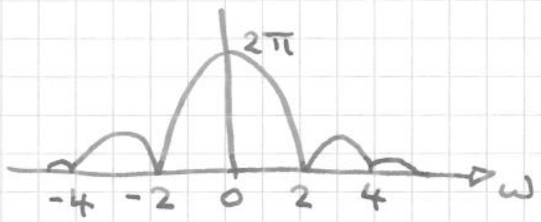

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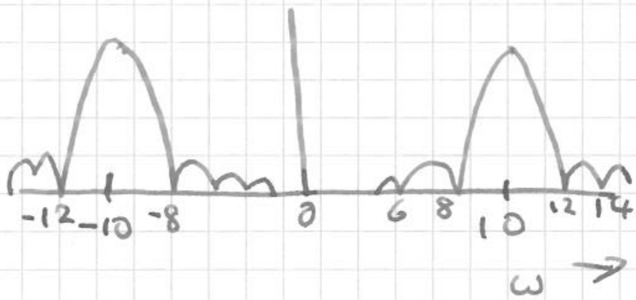


6. (b)

f(t) is simply shifted in time by 2π.

$\therefore$ FT is the same as (a), but multiplied by $e^{-j\omega(2\pi)}$.

$$\angle F(\omega) = -2\pi\omega \quad \text{(ie. linear phase)}.$$

$$F(\omega) = \pi \left\{ \text{sinc}^2\left[\frac{\pi(\omega-10)}{2}\right] + \text{sinc}^2\left[\frac{\pi(\omega+10)}{2}\right] \right\} e^{-j2\pi\omega}$$

//

(c)

f(t) is the same as (b) except the $\Delta\left(\frac{t}{2\pi}\right)$ is replace by $\text{rect}\left(\frac{t}{2\pi}\right)$.

$$\text{rect}\left(\frac{t}{2\pi}\right) \Longleftrightarrow 2\pi\,\text{sinc}(\pi\omega).$$

$$\therefore F(\omega) = \pi \left\{ \text{sinc}\left[\pi(\omega-10)\right] + \text{sinc}\left[\pi(\omega+10)\right] \right\} e^{-j2\pi\omega}$$

//

8. This is Lathi's book, Chap 7, Example 7.9 on p. 695.

$$U(\omega) = \int_{-\infty}^{\infty} u(t) e^{-j\omega t} dt = \int_{0}^{\infty} u(t) e^{-j\omega t} dt$$

$$= -\frac{1}{j\omega} e^{-j\omega t} \Big|_{0}^{\infty}$$

As $t \to \infty$, $e^{-j\omega t}$ gives an indeterminate answer.

Let us consider $u(t)$ as $e^{-at} u(t)$, $a \to 0$.

i.e. $u(t) = \lim_{a \to 0} e^{-at} u(t)$.

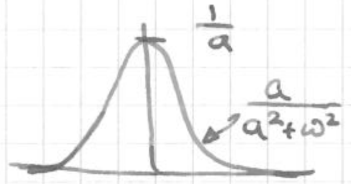
$$U(\omega) = \lim_{a \to 0} \mathcal{F}\{e^{-at} u(t)\} = \lim_{a \to 0} \frac{1}{a + j\omega}$$

Now divide this into real & imaginary parts.

$$U(\omega) = \lim_{a \to 0} \left[\frac{a}{a^2 + \omega^2}\right] - \lim_{a \to 0} \left[j \frac{\omega}{a^2 + \omega^2}\right]$$

$$= \lim_{a \to 0} \left[\frac{a}{a^2 + \omega^2}\right] + \frac{1}{j\omega}$$

Interesting function.



$$\int_{-\infty}^{\infty} \frac{a}{a^2 + \omega^2} d\omega = \tan^{-1} \frac{\omega}{a} \Big|_{-\infty}^{\infty} = \pi$$

and $a \to 0$, $\frac{a}{a^2 + \omega^2} = 0$ when $\omega \neq 0$, becomes an impulse at $\omega = 0$ with area (strength) π

$$\therefore U(\omega) = \pi \delta(\omega) + \frac{1}{j\omega} \quad //.$$